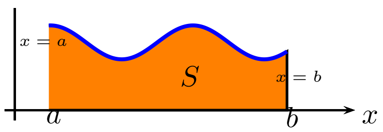
5.2 The Definite Integral

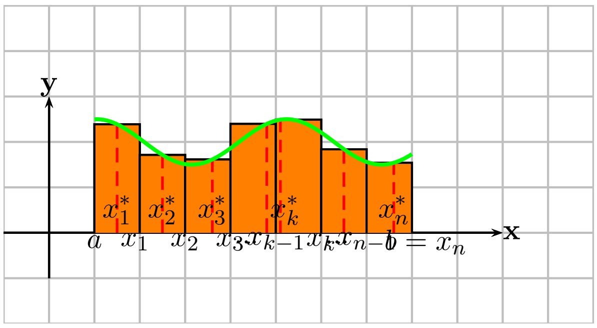
February 7, 2015

Definition 1: [Definition of A Definite Integral]

If $f(x)$ is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subinterval with length $\triangle x = \dfrac{b-a}{n}$. We let $a = x_0, x_2, \ldots, x_{k-1}, x_k, \ldots, x_{n-1}, x_n = b$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, x_3^*, \ldots, x_k^*, \ldots, x_n^*$ be any sample points in these subintervals, so $x_i^* \in [x_{i-1}, x_i]$. Then the definite integral of f from a to b is

$\int\limits_a^b f(x)dx = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i^*)\triangle x$ provided that this limit exists. If it does exist we say that f is integrable on $[a, b]$.





Note 1:

- The symbol $\int$ is called an integral sign. In the notation $\int\limits_a^b f(x)dx$ $f(x)$ is called integrand and a and b are called limits of integration, lower and upper limits respectively. The dx has no meaning by itself except indicates that the independent variable is x.

- The definite integral $\int\limits_a^b f(x)dx$ is a number, does not depend on x.

$$\int\limits_a^b f(x)dx = \int\limits_a^b f(t)dt = \int\limits_a^b f(y)dy.$$

- The sum $\sum\limits_{i=1}^{n} f(x_i^*)\triangle x$ is called a Riemann sum.

Theorem1: *[]*

If f is continuous on$[a,b]$ or if f has only a finite number of jump discontinuities, then f is integrable on $[a,b]$ That is the definite integral $\int\limits_a^b f(x)dx$ exists.

Theorem2: *[]*

If f is continuous on$[a,b]$, then

$$\int\limits_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\triangle x,$$

where $\triangle x = \dfrac{b-a}{n}$ and $x_i = a + i\triangle x$

Example 1: Express $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}(x_i^3 + x_i\sin x_i)\triangle x$, as an integral on $[0,\pi]$.

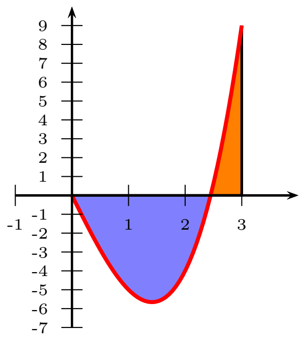
Solution:

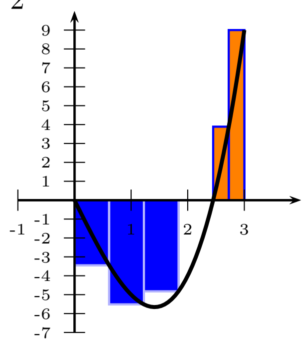
$\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}(x_i^3 + x_i\sin x_i)\triangle x = \int\limits_0^\pi (x^3 + x\sin x)\,dx$

Example 2: Evaluate the Riemann sum for $f(x) = x^3 - 6x$ taking the sample point to be the right endpoints and $a = 0$, $b = 3$, and $n = 6$.

Solution:

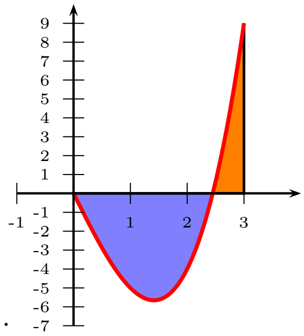
With $n = 6$, we have $\triangle x = \dfrac{3-0}{6} = \dfrac{1}{2}$ and $x_i = 0 + \dfrac{i}{2}$, $i = 1,2,3,4,5,6$. $R_6 = \sum\limits_{i=1}^{6} f(x_i)\triangle x = [f(0) + f(0.5) + f(1) + f(1.5) + f(2) + f(2.5) + f(3)]\dfrac{1}{2} = -3.9375$





Example 3: Evaluate $\int\limits_0^3 (x^3 - 6x)dx$.

Solution:



We have $\triangle x = \frac{3-0}{n} = \frac{3}{n}$ and $x_i = 0 + i\frac{3}{n} = \frac{3i}{n}$, $i = 1, 2, \ldots, n$. We approximate R_n.

$$\begin{aligned}\int_0^3 (x^3 - 6x)dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\triangle x = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]\frac{3}{n}\\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{27i^3}{n^3} - \frac{18i}{n}\right]\frac{3}{n}\\ &= \lim_{n\to\infty}\left[\frac{81}{n^4}\sum_{i=1}^{n} i^3 - \frac{54}{n^2}\sum_{i=1}^{n} i\right]\\ &= \lim_{n\to\infty}\left[\frac{81}{n^4}\left(\frac{n(n+1)}{2}\right)^2 - \frac{54}{n^2}\frac{n(n+1)}{2}\right]\\ &= \lim_{n\to\infty}\left[\frac{81}{4}\frac{n^2(n+1)^2}{n^4} - \frac{54}{2}\frac{n(n+1)}{n^2}\right]\\ &= \lim_{n\to\infty}\left[\frac{81}{4}\left(1+\frac{1}{n}\right)^2 - 27\left(1+\frac{1}{n}\right)\right] = \frac{81}{4} - 27 = \frac{-27}{4} = -6.75\end{aligned}$$

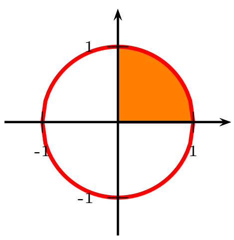
Example 4: Evaluate the following integrals by interpreting each in terms of area.

1. $\int_0^1 \sqrt{1-x^2}\, dx.$
2. $\int_0^3 (x-1)dx.$

Solution:

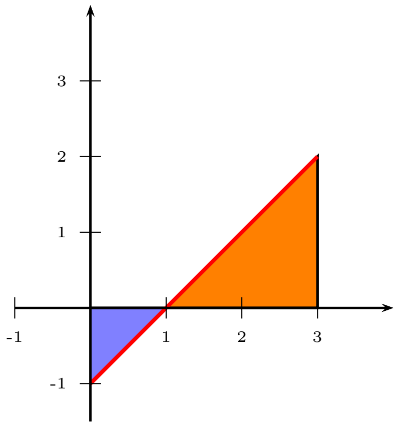
(1) Since $f(x) = \sqrt{1-x^2} \geq 0$, then the integral is the area under the curve $y = \sqrt{1-x^2}$ for $x = 0$ to $x = 1$. Now, since $y^2 = (\sqrt{1-x^2})^2 = 1 - x^2$ then $x^2 + y^2 = 1$ for $0 \leq x \leq 1$, and $y \geq 0$. This is quarter-circle with radius 1.

Therefore $\int_0^1 \sqrt{1-x^2}\, dx = \frac{1}{4}(\pi(1)^2) = \frac{\pi}{4}.$



(2) The integral represent the net area under the curve $y = x - 1$ for $0 \leq x \leq 3$. The area is the deference of the areas of the two triangle.

$$\int_0^3 (x-1)dx = A_1 - A_2 = \frac{1}{2}(2.2) - \frac{1}{2}(1.1) = \frac{3}{2}.$$



5.2 Properties of The Integral

1. $\int_a^b c\,dx = c(b-a)$ where c is any constant
2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ where c is any constant
4. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

5.3 Properties of The Integral

5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $a \leq c \leq b$
6. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$
7. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$
8. If $m \geq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \geq \int_a^b f(x)\,dx \leq M(b-a)$

Example 5: Use the properties of integrals to evaluate $\int_0^1 (4 + 3x^2)\,dx$.

Solution:

Note that $\int_0^1 4\,dx = 4(1-0) = 4$, and $\int_0^1 x^2\,dx = \frac{1}{3}$.

$\int_0^1 (4+3x^2)\,dx = \int_0^1 4\,dx + 3\int_0^1 x^2\,dx = 4 + 3\frac{1}{3} = 4+1 = 5.$

Example 6: If it is known that $\int_0^{10} f(x)\,dx = 17$ and $\int_0^{8} f(x)\,dx = 12$, find $\int_8^{10} f(x)\,dx$.

Solution:

Note that $\int_0^{8} f(x)\,dx + \int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx$

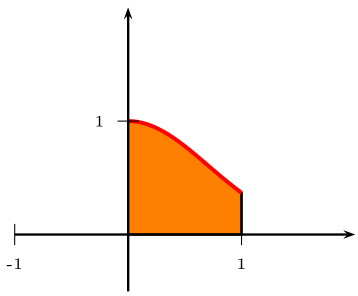
Hence $\int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx - \int_0^{8} f(x)\,dx = 17 - 12 = 5..$

Example 7: Use the properties of integrals to estimate $\int_0^1 e^{-x^2}\,dx$.

Solution:

Since $f(x) = e^{-x^2}$ is continuous on $[0,1]$ and differentiable on $(0,1)$, we can find the absolute maximum and absolute minimum. $f'(x) = -2xe^{-x^2} \Rightarrow f'(x) = 0 \Leftrightarrow x = 0$. Now, $f(0) = 1$, and $f(1) = \frac{1}{e}$, hence $\frac{1}{e} \leq e^{-x^2} \leq 1$, for $0 \leq x \leq 1$.

Then $\frac{1}{e}(1-0) \leq \int_0^1 e^{-x^2}\,dx \leq 1(1-0)$. Thus $\frac{1}{e} \leq \int_0^1 e^{-x^2}\,dx \leq 1$.



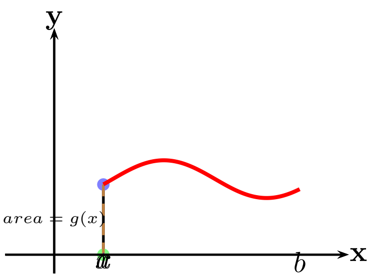
5.3 The Fundamental Theorem Of Calculus

February 7, 2015

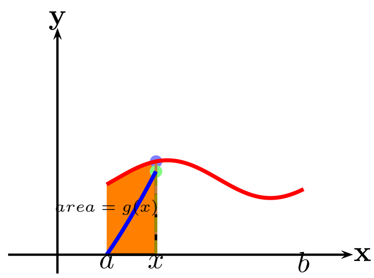
3 The Fundamental Theorem Of Calculus

3.1 The Fundamental Theorem Of Calculus, Part I

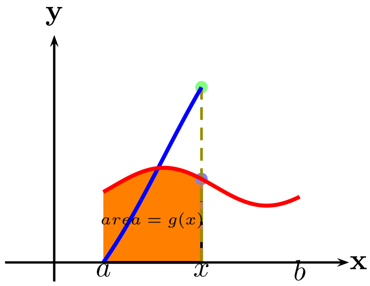
The first part of the Fundamental Theorem deals with a function defined by an integral " the function of net area under the curve of another function" given by $g(x) = \int_a^x f(t)\,dt$ where f is a continuous function on $[a,b]$ and x varies between a and b.



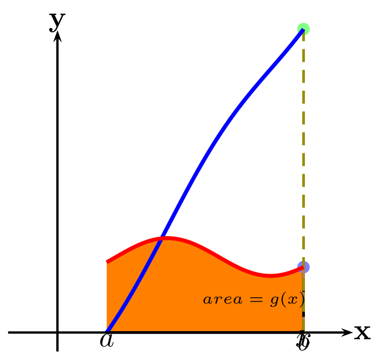
x	$f(x)$	$g(x)$



x	$f(x)$	$g(x)$



x	$f(x)$	$g(x)$



x	$f(x)$	$g(x)$

Theorem1: [The Fundamental Theorem of Calculus, Part I]

If f is continuous on $[a,b]$, then the function g defined by $g(x) = \int_a^x f(t)\,dt \quad a \le x \le b$ is continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) = f(x)$.

Note 1: Using Leibeniz notation for derivatives, we can write FTC1 as $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$. Also

(1) $\frac{d}{dx}\int_a^{h(x)} f(t)\,dt = f(h(x))h'(x)$

(2) $\frac{d}{dx}\int_{h(x)}^{a} f(t)\,dt = -\frac{d}{dx}\int_a^{h(x)} f(t)\,dt = -f(h(x))h'(x)$

(3) $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x)$

Example 1: Find the derivative of $\int_0^x \sqrt{1+t^2}\,dt$.

Solution:

Using FTC1, $\frac{d}{dx}\int_0^x \sqrt{1+t^2}\,dt = \sqrt{1+x^2}$.

Example 2: Find $\frac{d}{dx}\int_0^{x^4} \sec t\,dt$.

Solution:

Using the note after FTC1, $\frac{d}{dx}\int_0^{x^4} \sec t\,dt = \sec(x^4)4x^3 = 4x^3\sec(x^4)$.

Another solution: let $u = x^4$, then $\frac{d}{dx}\int_0^{x^4} \sec t\,dt = \frac{d}{du}\int_0^{u} \sec t\,dt\frac{du}{dx} = \sec u\frac{du}{dx} = 4x^3\sec(x^4)$.

Theorem2: [The Fundamental Theorem of Calculus, Part II]

If f is continuous on $[a,b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$ where F is any antiderivative of f. (A function F such that $F'(x) = f(x)$.)

Note 2: We usually use the following notation $\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a)$.

Example 3: Evaluate $\int_0^3 e^x\,dx$.

Solution:

Since $(e^x)' = e^x$, then an antiderivative of e^x is e^x.

Hence $\int_0^3 e^x\,dx = e^x|_0^3 = e^3 - e^0 = e^3 - 1$.

Example 4: Find the area under the curve $y = x^2$ from 0 to 1.

Solution:

Since $\left(\frac{x^3}{3}\right)' = x^2$, then an antiderivative of x^2 is $\frac{x^3}{3}$.

Hence $A = \int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$.

Example 5: Evaluate $\int_3^6 \frac{1}{x}\,dx$.

Solution:

Since $(\ln|x|)' = \frac{1}{x}$, then an antiderivative of $\frac{1}{x}$ is $\ln|x|$.

Hence $\int_3^6 \frac{1}{x}\,dx = \ln|x||_3^6 = \ln 6 - \ln 3 = \ln 2$.

Example 6: Find the area under the curve $y = \cos x$ from 0 to b where $0 \le b \le \frac{\pi}{2}$.

Solution:

Since $(\sin x)' = \cos x$, then an antiderivative of $\cos x$ is $\sin x$.

Hence $A = \int_0^b \cos x\,dx = \sin x|_0^b = \sin b - \sin 0 = \sin b$.

Example 7: What is wrong with the following $\int_{-1}^3 \frac{1}{x^2}\,dx = \left.\frac{-1}{x}\right|_{-1}^3 = \frac{-1}{3} - 1 = \frac{-4}{3}$.

Solution:

Notice that $\frac{1}{x^2} > 0$, and hence the integral can not be negative. Since $\frac{1}{x^2}$, is discontinuous on $[-1, 3]$, then we can not use the FTOC2 for this function.

Theorem3: [The Fundamental Theorem of Calculus]

Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.
2. $\int_a^b f(x)\,dx = F(b) - F(a)$ where F is any antiderivative of f. (A function F such that $F'(x) = f(x)$.)

5.4 Indefinite Integrals and The Net Change Theorem

February 7, 2015

4 Indefinite Integrals and The Net Change Theorem

The notation $\int f(x)\,dx$ used for antiderivative of f and is called an indefinite integral.

Thus $\int f(x)\,dx = F(x)$ means $F'(x) = f(x)$

1. $\int cf(x)\,dx = c\int f(x)\,dx$	2. $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
3. $\int k\,dx = kx + C$	4. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$
5. $\int e^x\,dx = e^x + C$	6. $\int a^x\,dx = \frac{a^x}{\ln a} + C$
7. $\int \frac{1}{x}\,dx = \ln\|x\| + C$	8. $\int \sin x\,dx = -\cos x + C$
9. $\int \cos x\,dx = \sin x + C$	10. $\int \sec^2 x\,dx = \tan x + C$
11. $\int \sec x \tan x\,dx = \sec x + C$	12. $\int \csc x \cot x\,dx = -\csc x + C$
13. $\int \csc^2 x\,dx = -\cot x + C$	14. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$
15. $\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C$	16. $\int \cosh x\,dx = \sinh x + C$
17. $\int \sinh x\,dx = \cosh x + C$	

Example 1: Find $\int (10x^4 - 2\sec^2 x)\, dx$.

Solution:

$$\begin{aligned}\int (10x^4 - 2\sec^2 x)\, dx &= 10\int x^4\, dx - 2\int \sec^2 x\, dx \\ &= 10\frac{x^5}{5} - 2\tan x + C \\ &= 2x^5 - 2\tan x + C.\end{aligned}$$

Example 2: Find $\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$.

Solution:

$$\begin{aligned}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta &= \int \frac{1}{\sin\theta}\frac{\cos\theta}{\sin\theta}\, d\theta \\ &= \int \cot\theta\csc\theta\, d\theta \\ &= -\csc\theta + C.\end{aligned}$$

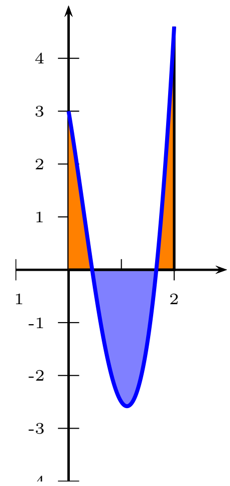
Example 3: Find $\int_0^3 (x^3 - 6x)\, dx$.

Solution:

$$\begin{aligned}\int_0^3 (x^3 - 6x)\, dx &= \frac{x^4}{4} - 6\frac{x^2}{2}\Big]_0^3 \\ &= \frac{x^4}{4} - 3x^2\Big]_0^3 \\ &= \frac{3^4}{4} - 3(3)^2 - 0 \\ &= \frac{81}{4} - 27 = \frac{-27}{4}.\end{aligned}$$

Example 4: Find $\displaystyle\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right) dx$.

Solution:



$$\begin{aligned}\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right) dx &= 2\int_0^2 x^3\,dx - 6\int_0^2 x\,dx + 3\int_0^2 \frac{1}{x^2+1}\,dx \\ &= 2\frac{x^4}{4} - 6\frac{x^2}{2} + 3\tan^{-1}x\Big]_0^2 \\ &= \frac{x^4}{2} - 3x^2 + 3\tan^{-1}x\Big]_0^2 \\ &= \frac{16}{2} - 3(2)^2 + 3\tan^{-1}2 \\ &= -4 + 3\tan^{-1}2 \approx -0.67855\end{aligned}$$

Example 5: Find $\displaystyle\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt$.

Solution:

$$\begin{aligned}\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt &= \int_1^9 \frac{2t^2}{t^2} + \frac{t^2\sqrt{t}}{t^2} - t^{-2}\,dt \\ &= \int_1^9 2 + t^{1/2} - t^{-2}\,dt \\ &= 2t + \frac{2}{3}t^{3/2} + \frac{1}{t}\Big]_9^1 \\ &= \left[18 + 18 + \frac{1}{9}\right] - \left[2 + \frac{2}{3} + 1\right] \\ &= 36 - 3 + \frac{1}{9} - \frac{6}{9} \\ &= 33 - \frac{5}{9} = 32 + \frac{4}{9} = 32\frac{4}{9}.\end{aligned}$$

Theorem1: [The Net Change Theorem]

The integral of a rate of change is the net change:

$\displaystyle\int_a^b F'(x)\,dx = F(b) - F(a)$. ***Note 1:***

- If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, hence $\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$ is the net change of position, or displacement.

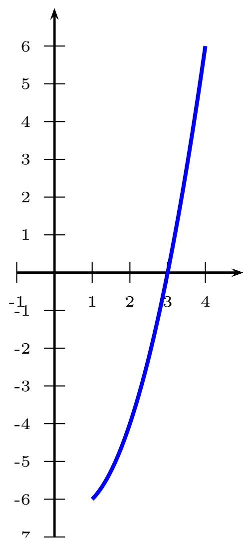
- The speed is given by $|v(t)|$, hence the total distance traveled is $\int_{t_1}^{t_2} |v(t)|\,dt$.
- The acceleration of an object is $a(t) = v'(t)$, so $\int_{t_1}^{t_2} a(t)\,dt = v(t_2) - v(t_1)$ is the change in velocity from t_1 to t_2.

Example 6: A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6 \quad m/s$.

1. Find the displacement of the particle on the interval $[1, 4]$.
2. Find the distance traveled on the interval $[1, 4]$.

Solution:

The velocity function $v(t) = t^2 - t - 6$ on the interval $[1, 4]$ has this graph.



1.

$$s(4) - s(1) = \int_1^4 t^2 - t - 6\,dt$$
$$= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4 = \frac{-9}{2}.$$

2. Note that $v(t) \leq 0$ on $[1, 3]$ and $v(t) \geq 0$ on $[3, 4]$.

$$\int_1^4 |v(t)|\,dt = \int_1^3 -(t^2 - t - 6)\,dt + \int_3^4 t^2 - t - 6\,dt$$
$$= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4 = \frac{61}{6}.$$

5.5 The Substitution Rule

February 7, 2015

5 The Substitution Rule

Let f be integrable function.

1. $\int (f(x))^n f'(x)dx = \frac{(f(x))^{n+1}}{n+1} + C \quad$ for $n \neq -1$

2. $\int \sin(f(x)) f'(x)dx = -\cos(f(x)) + C$

3. $\int \cos(f(x)) f'(x)dx = \sin(f(x)) + C$

4. $\int \sec^2(f(x)) f'(x)dx = \tan(f(x)) + C$

5. $\int \csc^2(f(x)) f'(x)dx = -\cot(f(x)) + C$

6. $\int \csc(f(x)) \cot(f(x)) f'(x)dx = -\csc(f(x)) + C$

7. $\int \sec(f(x)) \tan(f(x)) f'(x)dx = \sec(f(x)) + C$

8. $\int a^{f(x)} f'(x)\,dx = \frac{a^{f(x)}}{\ln a} + C$

9. $\int e^{f(x)} f'(x)\,dx = e^{f(x)} + C$

10. $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

11. $\int \frac{f'(x)}{\sqrt{1-[f(x)]^2}}\,dx = \sin^{-1}(f(x)) + C$

12. $\int \frac{f'(x)}{[f(x)]^2+1}\,dx = \tan^{-1}(f(x)) + C$

13. $\int \cosh(f(x)) f'(x)\,dx = \sinh(f(x)) + C$

14. $\int \sinh(f(x)) f'(x)\,dx = \cosh(f(x)) + C$

Example 1: Find $\int x^3 \cos(x^4+2)\,dx$.

Solution:

Using change of variable,

let $u = x^4 + 2$, then $du = 4x^3 dx$.

Thus $x^3 dx = \frac{du}{4}$. Hence

$$\begin{aligned}\int x^3 \cos(x^4+2)\,dx &= \int \cos(x^4+2)x^3 dx\\ &= \int \cos u \frac{du}{4}\\ &= \frac{1}{4}\int \cos u\,du\\ &= \frac{1}{4}\sin u + C\\ &= \frac{1}{4}\sin(x^4+2) + C\end{aligned}$$

Direct integration,

$$\begin{aligned}\int x^3 \cos(x^4+2)\,dx &= \frac{1}{4}\int \cos(x^4+2)\,4x^3 dx\\ &= \frac{1}{4}\int \underbrace{\cos(x^4+2)}_{\cos(f(x))}\underbrace{4x^3 dx}_{f'(x)dx}\\ &= \frac{1}{4}\underbrace{\sin(x^4+2)}_{\sin(f(x))} + C\end{aligned}$$

Example 2: Find $\int \sqrt{2x+1}\,dx$.

Solution:

Using change of variable,

let $u = 2x + 1$, then $du = 2dx$.

Thus $dx = \frac{du}{2}$. Hence

$$\begin{aligned}\int \sqrt{2x+1}\,dx &= \int \sqrt{u}\frac{du}{2}\\ &= \frac{1}{2}\int u^{1/2}\,du\\ &= \frac{1}{2}\frac{u^{3/2}}{3/2} + C\\ &= \frac{1}{\not 2}\frac{\not 2}{3}\sqrt{u^3} + C\\ &= \frac{1}{3}\sqrt{(2x+1)^3} + C\end{aligned}$$

Direct integration,

$$\begin{aligned}\int \sqrt{2x+1}\,dx &= \frac{1}{2}\int \sqrt{2x+1}\,2dx\\ &= \frac{1}{2}\int \underbrace{(2x+1)^{1/2}}_{[f(x)]^n}\ \underbrace{2dx}_{f'(x)dx}\\ &= \frac{1}{\not 2}\underbrace{\frac{(2x+1)^{3/2}}{3/\not 2}}_{\frac{[f(x)]^{n+1}}{n+1}} + C\\ &= \frac{1}{3}\sqrt{(2x+1)^3} + C\end{aligned}$$

Example 3: Find $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx$.

Solution:

Using change of variable,

let $u = 1-4x^2$, then $du = -8x dx$.

Thus $x dx = \dfrac{-du}{8}$. Hence

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = \int \frac{1}{\sqrt{1-4x^2}}\,x dx$$
$$= \int \frac{1}{\sqrt{u}}\frac{-du}{8}$$
$$= \frac{-1}{8}\int u^{-1/2}\,du$$
$$= \frac{-1}{8}\frac{u^{1/2}}{1/2} + C$$
$$= \frac{-2}{8}\sqrt{u} + C$$
$$= \frac{-1}{4}\sqrt{1-4x^2} + C$$

Direct integration,

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = \frac{-1}{8}\int (1-4x^2)^{-1/2}\,{-8}x dx$$
$$= \frac{-1}{8}\int \underbrace{(1-4x^2)^{-1/2}}_{[f(x)]^n}\;\underbrace{-8x dx}_{f'(x)dx}$$
$$= \frac{-1}{8}\underbrace{\frac{(1-4x^2)^{1/2}}{1/2}}_{\frac{[f(x)]^{n+1}}{n+1}} + C$$
$$= \frac{-2}{8}\sqrt{1-4x^2} + C$$
$$= \frac{-1}{4}\sqrt{1-4x^2} + C.$$

Example 4: Find $\displaystyle\int e^{5x}\,dx$.

Solution:

Using change of variable,

let $u = 5x$, then $du = 5dx$.

Thus $dx = \dfrac{du}{5}$. Hence

$$\int e^{5x}\,dx = \int e^u \frac{du}{5}$$
$$= \frac{1}{5}\int e^u\,du$$
$$= \frac{1}{5}e^u + C$$
$$= \frac{1}{5}e^{5x} + C$$

Direct integration,

$$\int e^{5x}\,dx = \frac{1}{5}\int e^{5x}\,5dx$$
$$= \frac{1}{5}\int \underbrace{e^{5x}}_{e^{f(x)}}\;\underbrace{5dx}_{f'(x)dx}$$
$$= \frac{1}{5}\underbrace{e^{5x}}_{e^{f(x)}} + C$$
$$= \frac{1}{5}e^{5x} + C$$

Example 5: Calculate $\displaystyle\int \tan x\, dx$.

Solution:

Note that $\displaystyle\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$.

Using change of variable,

let $u = \cos x$, then $du = -\sin x dx$.

Thus $\sin x dx = -du$. Hence

$$\begin{aligned}\int \tan x\, dx &= \int \frac{\sin x}{\cos x}\, dx\\ &= \int \frac{1}{\cos x} \sin x dx\\ &= \int \frac{1}{u}(-du)\\ &= -\int \frac{1}{u}\, du\\ &= -\ln|u| + C\\ &= \ln|u|^{-1} + C\\ &= \ln|\frac{1}{\cos x}| + C\\ &= \ln|\sec x| + C\end{aligned}$$

Direct integration,

$$\begin{aligned}\int \tan x\, dx &= \int \frac{\sin x}{\cos x}\, dx\\ &= -1 \int \frac{\overbrace{-1 \sin x}^{f'(x)}}{\cos x}\, dx\\ &= -\int \frac{-\sin x}{\underbrace{\cos x}_{f(x)}} dx\\ &= \underbrace{-\ln(|\cos x|)}_{\ln|f(x)|} + C\\ &= \ln|\cos x|^{-1} + C\\ &= \ln|\frac{1}{\cos x}| + C\\ &= \ln|\sec x| + C.\end{aligned}$$

Example 6: Calculate $\displaystyle\int \cot x\, dx$.

Solution:

Note that $\displaystyle\int \cot x\, dx = \int \frac{\cos x}{\sin x}\, dx$.

Using change of variable,

let $u = \sin x$, then $du = \cos x dx$.

Thus $\cos x dx = du$. Hence

$$\begin{aligned}\int \cot x\, dx &= \int \frac{\cos x}{\sin x}\, dx\\ &= \int \frac{1}{\sin x} \cos x dx\\ &= \int \frac{1}{u}\, du\\ &= \ln|u| + C\\ &= \ln|\sin x| + C\end{aligned}$$

Direct integration,

$$\begin{aligned}\int \cot x\, dx &= \int \frac{\overbrace{\cos x}^{f'(x)}}{\sin x}\, dx\\ &= \int \frac{\cos x}{\underbrace{\sin x}_{f(x)}} dx\\ &= \underbrace{\ln(|\sin x|)}_{\ln|f(x)|} + C\\ &= \ln|\sin x| + C\end{aligned}$$

Example 7: Find $\displaystyle\int_0^4 \sqrt{2x+1}\, dx$.

Solution:

Using change of variable,

let $u = 2x + 1$, then $du = 2dx$.

Thus $dx = \frac{du}{2}$.

when $x = 0 \Rightarrow u = 2(0) + 1 = 1$ and

$x = 4 \Rightarrow u = 2(4) + 1 = 9$. Hence

$$\int_0^4 \sqrt{2x+1}\,dx = \int_1^9 \sqrt{u}\frac{du}{2}$$
$$= \frac{1}{2}\int_1^9 u^{1/2}\,du$$
$$= \frac{1}{\cancel{2}}\frac{\cancel{2}}{3}\sqrt{u^3}\Big|_1^9$$
$$= \frac{1}{3}\sqrt{u^3}\Big|_1^9 \quad = \frac{26}{3}$$

Direct integration,

$$\int_0^4 \sqrt{2x+1}\,dx = \frac{1}{2}\int_0^4 \sqrt{2x+1}\,2dx$$
$$= \frac{1}{2}\int_0^4 \underbrace{(2x+1)^{1/2}}_{[f(x)]^n}\ \underbrace{2dx}_{f'(x)dx}$$
$$= \frac{1}{\cancel{2}}\underbrace{\frac{(2x+1)^{3/2}}{3/\cancel{2}}}_{\frac{[f(x)]^{n+1}}{n+1}}\Bigg|_0^4$$
$$= \frac{1}{3}\sqrt{(2x+1)^3}\Big|_0^4 \qquad = \frac{26}{3}$$

Example 8: Evaluate $\int_1^2 \frac{dx}{(3-5x)^2}$.

Solution:

Note $\int_1^2 \frac{dx}{(3-5x)^2} = \int_1^2 (3-5x)^{-2}\,dx$.

Using change of variable,

let $u = 3 - 5x$, then $du = -5dx \Rightarrow dx = \frac{-du}{5}$.

when $x = 1 \Rightarrow u = 3 - 5(1) = -2$ and

$x = 2 \Rightarrow u = 3 - 5(2) = -7$. Hence

$$\int_1^2 (3-5x)^{-2}\,dx = \int_{-2}^{-7} u^{-2}\,\frac{-du}{5}$$
$$= \frac{-1}{5}\int_{-2}^{-7} u^{-2}\,du$$
$$= -\frac{-1}{5}\int_{-7}^{-2} u^{-2}\,du$$
$$\frac{1}{5}(-u^{-1})\Big|_{-7}^{-2} = \frac{-1}{5u}\Big|_{-7}^{-2} = \frac{1}{14}$$

Direct integration,

$$\int_1^2 \frac{dx}{(3-5x)^2} = \frac{-1}{5}\int_1^2 (3-5x)^{-2}\,-5dx$$
$$= \frac{-1}{5}\int_1^2 \underbrace{(3-5x)^{-2}}_{[f(x)]^n}\underbrace{-5dx}_{f'(x)dx}$$
$$= \frac{-1}{5}\underbrace{\frac{(3-5x)^{-1}}{-1}}_{\frac{[f(x)]^{n+1}}{n+1}}\Bigg|_1^2$$
$$= \frac{1}{5(3-5x)}\Big|_1^2 = \frac{1}{14}$$

Example 9: Find $\int_1^e \frac{\ln x}{x}\,dx$.

Solution:

Note $\int_1^e \frac{\ln x}{x}\,dx = \int_1^e \ln x \frac{1}{x}\,dx$.

Using change of variable,

let $u = \ln x$, then $du = \frac{1}{x}dx$.

when $x = 1 \Rightarrow u = \ln 1 = 0$ and

$x = e \Rightarrow u = \ln e = 1$. Hence

$$\begin{aligned}\int_1^e \ln x \frac{1}{x}dx &= \int_0^1 u\,du \\ &= \frac{1}{2}u^2\Big|_0^1 \\ &= \frac{1}{2}[1^2 - 0^2] \\ &= \frac{1}{2}\end{aligned}$$

Direct integration,

$$\begin{aligned}\int_1^e \frac{\ln x}{x}\,dx &= \int_1^e \ln x \frac{1}{x}dx \\ &= \int_1^e \underbrace{\ln x}_{[f(x)]^n} \underbrace{\frac{1}{x}dx}_{f'(x)dx} \\ &= \underbrace{\frac{(\ln x)^2}{2}}_{\frac{[f(x)]^{n+1}}{n+1}}\Bigg|_1^e \\ &= \frac{(\ln e)^2}{2} - \frac{(\ln 1)^2}{2} \\ &= \frac{1}{2}\end{aligned}$$

Example 10: Find $\int x\sqrt{1+x}\,dx$.

Solution:

Note that $(1+x)' = 1$, and hence direct integration does not work.

Let $u = \sqrt{1+x} \Leftrightarrow u^2 = 1 + x \Leftrightarrow u^2 - 1 = x$. Hence $2udu = dx$.

$$\begin{aligned}\int x\sqrt{1+x}\,dx &= \int (u^2-1)u\,2udu \\ &= \int [2u^4 - 2u^2]\,du \\ &= \frac{2}{5}u^5 - \frac{2}{3}u^3 + C \\ &= \frac{2}{5}(\sqrt{x+1})^5 - \frac{2}{3}(\sqrt{x+1})^3 + C \\ &= \frac{2}{5}\sqrt{(x+1)^5} - \frac{2}{3}\sqrt{(x+1)^3} + C\end{aligned}$$

Example 11: Find $\int x^5\sqrt{1+x^2}\,dx$.

Solution:

Note that $(1+x^2)' = 2x$, and hence direct integration does not work.

Let $u = \sqrt{1+x^2} \Leftrightarrow u^2 = 1+x^2 \Leftrightarrow u^2 - 1 = x^2$. Hence $2udu = 2xdx$. Thus $xdx = udu$.

$$\begin{aligned}\int x^5\sqrt{1+x^2}\,dx &= \int x^4\sqrt{1+x^2}\,xdx \\ &= \int (u^2-1)^2 u\,udu \\ &= \int [u^4-2u^2+1]u^2\,du \\ &= \int [u^6-2u^4+u^2]\,du \\ &= \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C \\ &= \frac{1}{7}\sqrt{(x^2+1)^7} - \frac{2}{5}\sqrt{(x^2+1)^5} + \frac{1}{3}\sqrt{(x^2+1)^3} + C\end{aligned}$$

Example 12: Find $\int \frac{x^2}{\sqrt{1+x}}\,dx$.

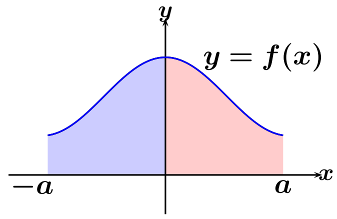
Solution:

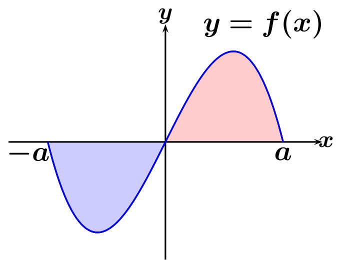
Note that $(1+x)' = 1$, and hence direct integration does not work.

Let $u = \sqrt{1+x} \Leftrightarrow u^2 = 1+x \Leftrightarrow u^2 - 1 = x$. Hence $2udu = dx$.

$$\begin{aligned}\int \frac{x^2}{\sqrt{1+x}}\,dx &= \int \frac{(u^2-1)^2}{\cancel{u}}\,2\cancel{u}du \\ &= \int 2[u^4-2u^2+1]\,du \\ &= 2[\frac{1}{5}u^5 - \frac{2}{3}u^3 + u] + C \\ &= \frac{2}{5}\sqrt{(x+1)^5} - \frac{4}{3}\sqrt{(x+1)^3} + 2\sqrt{x+1} + C\end{aligned}$$

Now a few words about the role of symmetry in integration of functions. Suppose that f is a continuous function defined on some interval of the form $[-a, a]$, some closed interval that is symmetric about the origin.





i) If f is an even [i.e. $f(-x) = f(x)$] on $[-a, a]$, then

$$\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx.$$

ii) If f is an odd [i.e. $f(-x) = -f(x)$] on $[-a, a]$, then

$$\int_{-a}^{a} f(x)\, dx = 0.$$

Example 13: Find $\displaystyle\int_{-2}^{2} \frac{\sin x}{x^4 + x^2 + 1}\, dx$.

Solution:

Let $f(x) = \dfrac{\sin x}{x^4 + x^2 + 1}$, then $f(-x) = \dfrac{\sin(-x)}{(-x)^4 + (-x)^2 + 1} = -\dfrac{\sin x}{x^4 + x^2 + 1} = -f(x)$. Hence f is odd function. $\displaystyle\int_{-2}^{2} \frac{\sin x}{x^4 + x^2 + 1}\, dx = 0$.

Example 14: Find $\displaystyle\int_{-3}^{3} (x^2 + 1)\, dx$.

Solution:

Let $f(x) = x^2 + 1$, then $f(-x) = (-x)^2 + 1 = x^2 + 1 = f(x)$. Hence f is even function.

$$\int_{-3}^{3} (x^2 + 1)\, dx = 2\int_{0}^{3} (x^2 + 1)\, dx = 2\left(\frac{x^3}{3} + x\right)\Big|_0^3 = 2\left[\frac{3^3}{3} + 3\right] = 24.$$

Example 15: If f is an even function and $\displaystyle\int_{-4}^{4} f(x)\, dx = 16$. Find $\displaystyle\int_{0}^{4} \frac{f(x)}{4}\, dx = 16$.

Solution:

Note that, since $f(x)$ is even, then $16 = \displaystyle\int_{-4}^{4} f(x)\, dx = 2\int_{0}^{4} f(x)\, dx$. Hence $\displaystyle\int_{0}^{4} f(x)\, dx = \frac{16}{2} = 8$.

$$\int_0^4 \frac{f(x)}{4}\,dx = \frac{1}{4}\int_0^4 f(x)\,dx = \frac{1}{4}8 = 2.$$